Quantum kinetic equation in the closed-time-path formalism

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A systematic derivation of the quantum kinetic equation is presented in the framework of a closed-time-path formalism. Introducing a probe, the expectation value of the number operator is calculated as a functional of the probing source. Then, solving for the source by inverting the relation, the removal of the source leads to the quantum kinetic equation as the equation of motion for the number, which gives a generalization of the Boltzmann equation including memory. The inversion formula is used in the course of the derivation. The calculation is presented up to third order in interaction, and the effect of initial correlations is also considered.

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I. INTRODUCTION

In this paper, we investigate the quantum kinetic equation (QKE) using the inversion method briefly reported in Ref. [1]. There has been a lot of work on the derivation of the QKE or the generalized Boltzmann equation, and the most popular approach is the generalized Kadanoff-Baym (GKB) formalism [2–4]. In these approaches, starting from the Dyson equation for a two-point Green function, the equation for the one-particle density is derived with the aid of the Kadanoff-Baym ansatz [2] or its generalized version [3,4]. Although this ansatz successfully leads to the QKE, the range of validity is not known with certainty.

An alternative approach using the Green function is the counterterm method based on the closed-time-path (CTP) formalism [5–7], or on the thermofield dynamics [8]. These approaches have the advantage that they do not require an ansatz for the expressions of the Green functions. Instead, a counterterm is first introduced into the CTP or thermofield Lagrangian, and an unperturbed propagator is calculated. Then to determine the counterterm, some condition, such as the cancellation of the on-shell part of the self-energy [5,6,8] or the cancellation of the pinch singularity [7], is adopted, which leads to the generalized Boltzmann equation. These conditions, however, are of course not uniquely determined, and ambiguity appears in this step.

In this paper, we use a more direct approach reported in Ref. [1] which overcomes these shortcomings. It is based on the inversion method [9,10] which is a systematic procedure to derive the equation of motion (EOM) in the CTP formalism; Coupling the probing source to the order parameter, which is chosen to be the number in this paper, the expectation value of the order parameter is first calculated under the existence of this external source. After this functional relation is inverted perturbatively for the source, the EOM of the expectation value is obtained when the source is removed. In this method, the expectation value of the number is directly calculated, and the QKE is derived as its EOM which has the form of a non-Markovian Boltzmann equation. Hence no ansatz is made in the course of deriving the EOM, and, in contrast to the counterterm method, there is no ambiguity in determining the EOM since we just remove the source after the inversion.

In addition to the calculation in Ref. [1], where only the next-to-leading order correction (second order in interaction) was considered, we proceed to a calculation up to third order in interaction, and the inclusion of an initial correlation effect is also discussed. For the higher-order correction, some new terms appear including the contribution of three-body collision. The effect of the initial correlation is considered by the usual method; extending the time-path to include imaginary time [11–13,4], and new initial correlation terms are found in addition to the terms derived in other works.

The paper is organized as follows. After a brief description of the inversion method in Sec. II B, we apply it to the case of number operator in Sec. III. As discussed in Ref. [1], the method is not naively applicable to the number operator, but this problem is solved by introducing a more efficient probing source for handling the number. Our choice of the probe is dictated by the structure of the dissipative counterterm [5], which is summarized in Appendix A. In Appendix B, another choice of the probe source is discussed. The higher-order correction is discussed in Sec. III D, and energy conservation with the obtained QKE is proved in Sec. III E. Then, in Sec. IV, the initial correlation effect on the QKE is calculated, and the stationarity of the initial equilibrium case is confirmed. Section V is devoted to discussions.

II. DERIVATION OF EOM IN CTP FORMALISM—INVERSION METHOD

Let us briefly describe the inversion method [9,10]. It is a systematic procedure to derive the EOM of the expectation value of an arbitrary operator, say $Q(\hat{\varphi})$, which is a function of the dynamical variable $\hat{\varphi}$ of the system. Introducing a probing external source J, we first derive the expectation value $Q(t)=\langle Q(\hat{\varphi}(t))\rangle$ as a functional of the source J: $Q(t)=f[t;J]$. Then, solving this inversely to express J as a functional of Q, $J(t)=g[t;Q]$, we remove the source J. The resultant equation $g[t;Q]=0$ determines the time dependence of Q, i.e., it gives the EOM.

A. CTP formalism

In the first step of calculating the expectation value, we utilize the CTP formalism [14,15]. In the CTP formalism, the

time dependent source is usually introduced in the following way. First, with the Hamiltonian $\hat{H}$ of $\hat{\varphi}$, the CTP generating functional W is defined as

$$e^{(i/\hbar)W[J_1,J_2]} \equiv \mathrm{Tr}\, T e^{-(i/\hbar)\int_{t_1}^{t_F} dt(\hat{H}-J_1(t)\hat{Q})}\hat{\rho}\tilde{T}e^{(i/\hbar)\int_{t_1}^{t_F} dt(\hat{H}-J_2(t)\hat{Q})} \tag{1}$$

$$\propto \int [d\varphi_1 d\varphi_2]\langle \varphi_{1I}|\hat{\rho}|\varphi_{2I}\rangle \times e^{(i/\hbar)\int_{t_I}^{t_F} dt(L(\varphi_1)-L(\varphi_2)+J_1Q(\varphi_1)-J_2Q(\varphi_2))}, \tag{2}$$

where $\hat{\rho}$ is the initial distribution, and T and $\tilde{T}$ are the time ordering and antiordering operators, respectively. The last equality is due to the path-integral representation, where φ_1 and φ_2 are introduced as integral variables along the forward and backward time branches, respectively. It can be seen from Eq. (1) that the expectation value of the products of Q can be obtained by a functional differentiation of W with respect to the source J_1 or J_2. The physically sensible situation is realized by setting $J_1=J_2=J$, since this gives the unitary time evolution; thereby, J plays the role of physical external source. Thus the expectation value of $\hat{Q}$ at time t under a physical external source J can be calculated as

$$Q(t)\equiv \left.\frac{\delta W[J_1,J_2]}{\delta J_1(t)}\right|_{J_1=J_2=J} = \langle \hat{Q}(t)\rangle_J. \tag{3}$$

This gives us the expectation value Q as a functional of physical probing source J. See, e.g., Refs. [14,15] for more detailed properties of generating functional.

B. Inversion formulas

In the second step of the inversion method, we solve relation (3) inversely to express J as a functional of Q. Then setting the external source $J=0$, the obtained relation gives the EOM of Q. Formally, the general expression of the EOM can be written with the Legendre transformation of W [10]. But practically, if the aim is to derive the EOM, the process of Legendre transformation is unnecessary, and this inversion can be carried out in the following perturbative fashion.

Usually Q as a functional of J, we obtain some perturbation series

$$Q(t)=f[t;J]=\sum_n \lambda^n f_n[t;J], \tag{4}$$

where λ is a small parameter, and $f[t;J]$ expresses that f is a function of t and a functional of J. Then if we write the inverted relation as

$$J(t)=g[t;Q]=\sum_m \lambda^m g_m[t;Q], \tag{5}$$

we obtain the simple identity

$$\begin{aligned} Q(t)&=f[t;g[Q]] \\ &=f_0[t;g_0[Q]] \\ &\quad+\lambda\left(\int ds f_0^{(1)}[t,s;Q]g_1[s;Q]+f_1[t;g_0[Q]]\right) \\ &\quad+\lambda^2\left(\int ds f_0^{(1)}[t,s;Q]g_2[s;Q]\right. \\ &\quad+\tfrac{1}{2}\int ds\, ds' f_0^{(2)}[t,s,s';Q]g_1[s;Q]g_1[s';Q] \\ &\quad\left.+\int ds f_1^{(1)}[t,s;Q]g_1[s;Q]+f_2[t;g_0[Q]]\right)+O(\lambda^3), \end{aligned} \tag{6}$$

where we have used the abbreviations

$$f_n(k)[t,s_1,s_2,\ldots,s_k;Q]\equiv \left.\frac{\delta^k f_n[t;J]}{\delta J(s_1)\delta J(s_2)\cdots\delta J(s_k)}\right|_{J=g_0[Q]}. \tag{7}$$

Comparing the left- and right-hand sides of Eq. (6) in each order of λ, we obtain the expressions for g_m in terms of f_n, which we call the ''inversion formulas'' [10]:

$$g_0[t;Q]=f_0^{-1}[t;Q], \tag{8}$$

$$g_1[t;Q]=-\int dt' f_0^{(1)^{-1}}[t,t';Q]f_1[t';g_0], \tag{9}$$

$$\begin{aligned} g_2[t;Q]=&-\int dt' f_0^{(1)^{-1}}[t,t';Q] \\ &\times\left(\tfrac{1}{2}\int ds ds' f_0^{(2)}[t',s,s';Q]g_1[s;Q]g_1[s';Q]\right. \\ &\left.+\int ds f_1^{(1)}[t',s;Q]g_1[s;Q]+f_2[t';g_0[Q]]\right), \end{aligned} \tag{10}$$

where $f_0^{(1)^{-1}}$ is defined by

$$\int dt' f_0^{(1)^{-1}}[t,t';Q]f_0^{(1)}[t',s;Q]=\delta(t-s). \tag{11}$$

Summarizing, we first calculate the expectation value as a functional of a physical external source in the framework of the CTP formalism, and then, by solving the functional inversely for the source with the aid of inversion formulas, the EOM of the expectation value is obtained by removing the external source. In the following, we apply this method to the case where Q is the number, and directly derive the EOM of the number which turns out to have the form of the QKE.

Note that the inversion method is nonperturbative in the following sense. As discussed in Ref. [10], even though the original series [Eq. (4)] is truncated and calculated with the finite number of diagrams, the inverted series [Eq. (5)] can include infinite diagrams through the process of inversion. Of course which subdiagrams are included in Q becomes different if we introduce the source in other way than Eq. (1), and hence the inverted series depend on the choice of the

source. A good source will handle Q efficiently, so that the inverted relation describes the dynamics of the expectation value in a satisfactory fashion. At least to make the inversion method work, we need a nontrivial lowest-order functional expression $f_0[t;J]$ which can be inversely solved for J. This becomes the key point for deriving the QKE.

III. DERIVATION OF QKE: GENERALIZED BOLTZMANN EQUATION

In this section, we first clarify the problem of applying the inversion method to the number operator. Then, in order to overcome the difficulty, a type of probing source is introduced, with which the inversion method works. The kinetic equation is then derived as an EOM of number in the next-to-leading order of the perturbation. A higher-order correction to the QKE is also presented.

A. Introduction of probing source

Let us see what is the problem in the case of the number operator. We consider a nonrelativistic boson field of a homogeneous system described by the Hamiltonian $\hat{H}=\hat{H}_0+\hat{H}_{\rm int}$, with

$$\hat{H}_0=\sum_{\mathbf{k}}\ \epsilon_{\mathbf{k}}\hat{\psi}^\dagger_{\mathbf{k}}\hat{\psi}_{\mathbf{k}}, \tag{12}$$

$$\hat{H}_{\rm int}=\frac{\lambda}{4}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\hat{\psi}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{\psi}^\dagger_{\mathbf{k}'-\mathbf{q}}\hat{\psi}_{\mathbf{k}}\hat{\psi}_{\mathbf{k}'}, \tag{13}$$

where $\hat{\psi}^\dagger_{\mathbf{k}}$ and $\hat{\psi}_k$ are creation and annihilation operators of momentum $\mathbf{k}$, respectively and λ is a coupling constant, which is assumed to be small and identified with the expansion parameter in Eq. (4). Extension to other type of interaction is straightforward. At an initial time t_1, the system is described by the density matrix $\hat{\rho}$, and we assume in this section that no initial correlation exists among different wave-number components; $\hat{\rho}$ can be written as a product from $\Pi_{\mathbf{k}}\hat{\rho}_{\mathbf{k}}$, where $\hat{\rho}_{\mathbf{k}}$ is a density matrix for each wave number which gives the expectation value of the number at initial time as $n_{\mathbf{k}}(t_{\rm I})=\mathrm{Tr}\,\hat{\rho}_{\mathbf{k}}\hat{\psi}^\dagger_{\mathbf{k}}\hat{\psi}_k$. Inclusion of the initial correlation is studied in Sec. IV.

In order to derive the EOM of the expectation value of the number $\hat{n}_{\mathbf{k}}(t)=\hat{\psi}^\dagger_{\mathbf{k}}(t)\hat{\psi}_{\mathbf{k}}(t)$, a naive choice of the source is to replace the Hamiltonian $\hat{H}$ by $\hat{H}-\Sigma_{\mathbf{k}}J_{\mathbf{k}}(t)\hat{\psi}^\dagger_{\mathbf{k}}(t)\hat{\psi}_{\mathbf{k}}(t)$. Then in a path-integral representation of the CTP generating functional given in Eq. (2), this source can be built into the free part of the Lagrangian as

$$L^J_0(\psi_1)-L^J_0(\psi_2)=\sum_{\mathbf{k}}\ \psi^*_{i,\mathbf{k}}\mathcal{D}_{ij,\mathbf{k}}\psi_{j,\mathbf{k}}, \tag{14}$$

with the matrix

$$\mathcal{D}_{\mathbf{k}}(t,\partial_t)\equiv\begin{pmatrix} i\hbar\partial_t-\epsilon_{\mathbf{k}}+J_{\mathbf{k}}(t) & 0 \\ 0 & -i\hbar\partial_t+\epsilon_{\mathbf{k}}-J_{\mathbf{k}}(t)\end{pmatrix}. \tag{15}$$

The unperturbed propagator is essentially the inverse of the matrix in Eq. (15), and, with this propagator, if we evaluate the expectation value $n_{\mathbf{k}}(t)$ in the absence of interaction, we obtain the initial value $\langle\hat{n}_{\mathbf{k}}(t)\rangle_J=n_{\mathbf{k}}(t_I)$. This is due to the fact that $\hat{H}_0-\Sigma_{\mathbf{k}}J_{\mathbf{k}}\hat{n}_{\mathbf{k}}$ commutes with $\hat{n}$; n is conserved at the level of free theory even when $J_{\mathbf{k}}\neq 0$. Since no dependence on J appears in $\langle\hat{n}_{\mathbf{k}}(t)\rangle_J$, we fail to make the lowest-order inversion corresponding to Eq. (8), and hence the inversion formulas cannot be used in this case. A probe of the form of Eq. (15) does not disturb the system so efficiently that the number operator cannot be handled.

Then why does the counterterm method in Ref. [5] work? According to Ref. [5], the time-local counterterm can be constructed so as to keep the following structure of the full propagator in the CTP formalism (we suppress the index of wave number for a while),

$$\begin{aligned} G(t,s)&\equiv-\mathrm{Tr}\,\hat{\rho}\begin{pmatrix} T\hat{\psi}(t)\hat{\psi}^\dagger(s) & \hat{\psi}^\dagger(s)\hat{\psi}(t) \\ \hat{\psi}(t)\hat{\psi}^\dagger(s) & \tilde{T}\hat{\psi}(t)\hat{\psi}^\dagger(s)\end{pmatrix}_c \\ &=\theta(t-s)\begin{pmatrix} h(t,s) & k(t,s) \\ h(t,s) & k(t,s)\end{pmatrix} \\ &\quad+\theta(s-t)\begin{pmatrix} k^*(s,t) & k^*(s,t) \\ h^*(s,t) & h^*(s,t)\end{pmatrix}, \end{aligned} \tag{16}$$

where the subscript c means the connected part, and

$$h(t,s)\equiv-\langle\hat{\psi}(t)\hat{\psi}^\dagger(s)\rangle_c, \tag{17}$$

$$k(t,s)\equiv-\langle\hat{\psi}^\dagger(s)\hat{\psi}(t)\rangle_c. \tag{18}$$

As far as structure (16) is kept, the perturbative calculation of the expectation value of any Hermitian operator yields the real value. Then, as shown in Appendix A, the counterterm $\psi^*_i\mathcal{M}_{ij}\psi_j$, with the matrix

$$\mathcal{M}(t)=\begin{pmatrix} \hbar\Delta\omega(t)-i\alpha(t) & -i(\hbar\gamma(t)-\alpha(t)) \\ i(\hbar\gamma(t)+\alpha(t)) & -\hbar\Delta\omega(t)-i\alpha(t)\end{pmatrix}, \tag{19}$$

is allowed to be subtracted from the free part of the Lagrangian. Here $\Delta\omega$, α, and γ are all real functions which are determined by appropriate conditions. The bare propagator calculated from $L_0(\psi_1)-L_0(\psi_2)-\psi^*_i\mathcal{M}_{ij}\psi_j$ leads to a nontrivial time dependence of the number in the absence of interaction. The existence of the parameters as a nondiagonal element in Eq. (19) is a crucial point.

Comparing Eqs. (19) and (15), the parameter we utilized as a physical external source in Eq. (15) corresponds to $\Delta\omega$ in Eq. (19). Equation (19), however, suggests that another physical source corresponding to α or γ can be introduced as a probe. Note that what the inversion method requires is the expression of the expectation value as a functional of some ''physically sensible'' source, and it is not restricted to a source of the form of Eq. (15). Here, ''physically sensible'' means that the expectation value of Hermitian operator under the existence of this source is guaranteed to be real.

Our choice in this paper is the source corresponding to α in Eq. (19). The source corresponding to γ can be treated

similarly, and is discussed in Appendix B. Then the free part of the Lagrangian including the source has now the matrix

$$\mathcal{D}(t,\partial_t)=\begin{pmatrix} i\hbar\partial_t-\epsilon+iJ(t) & -iJ(t) \\ -iJ(t) & -i\hbar\partial_t+\epsilon+iJ(t) \end{pmatrix}. \quad (20)$$

Note that although we have introduced the source as Eq. (20), what we will calculate in the following is just the expectation value of the number. For simplicity, we calculate the expectation value of the number by path integration using $\psi_1^*\psi_1$. But the results are the same for other choices; $\psi_2^*\psi_2, \psi_2^*\psi_1$, or their linear combinations. Note also that $\psi_1^*\psi_1$ or $\psi_2^*\psi_2$ should be understood as the product of field variables whose time arguments differ infinitesimally as $\psi_1^*(t+0)\psi_1(t)$ or $\psi_2^*(t-0)\psi_2(t)$, respectively, which results from the coherent-state path-integral representation.

B. Unperturbed propagator and number

Starting with Eq. (20), the unperturbed propagator G_0 is calculated from the relation

$$\mathcal{D}(t,\partial_t)G_0(t,s)=G_0(t,s)\mathcal{D}(s,-\tilde{\partial}_s) \quad (21)$$

$$=-i\hbar\,\delta(t-s), \quad (22)$$

where $\tilde{\partial}$ implies left differentiation. Since $\mathcal{D}$ has been chosen as to keep structure (16) unchanged, G_0 has the same structure in which h and k are replaced by h_0 and k_0, respectively. Then Eq. (22) leads to the equations

$$(i\hbar\partial_t-\epsilon)h_0(t,s)=0, \quad (23)$$

$$(i\hbar\partial_t-\epsilon)k_0(t,s)=0, \quad (24)$$

for $t>s$, and

$$(i\hbar\partial_t-\epsilon+iJ(t))k_0^*(s,t)=iJ(t)h_0^*(s,t), \quad (25)$$

$$(i\hbar\partial_t-\epsilon-iJ(t))h_0^*(s,t)=-iJ(t)k_0^*(s,t) \quad (26)$$

for $s>t$. The boundary conditions at $t=s$ are given as

$$h_0(s,s)-k_0^*(s,s)=-1, \quad k_0(s,s)-h_0^*(s,s)=1, \quad (27)$$

$$h_0(s,s)-h_0^*(s,s)=0, \quad k_0(s,s)-k_0^*(s,s)=0. \quad (28)$$

From Eqs. (28), $h_0(s,s)$ and $k_0(s,s)$ are real functions. Then two conditions in Eq. (27) are identical, and simply express the fact that the expectation value of the equal-time commutator $[\hat{\psi},\hat{\psi}^\dagger]$ is unity. Note that from definition (18), $k_0(t,t)$ gives the expectation value of the number operator (multiplied by -1) in the absence of the interaction. Which we denote as $n^{(0)}(t)$.

From Eqs. (23) and (24), we obtain, for $t>s$,

$$k_0(t,s)=e^{-(i/\hbar)\epsilon(t-s)}k_0(s,s)=-n^{(0)}(s)e^{-(i/\hbar)\epsilon(t-s)}, \quad (29)$$

$$h_0(t,s)=e^{-(i/\hbar)\epsilon(t-s)}h_0(s,s)=-(n^{(0)}(s)+1)e^{-(i/\hbar)\epsilon(t-s)}. \quad (30)$$

Then exchanging t and s in Eqs. (29) and (30) and taking the complex conjugation, $h_0^*(s,t)$ and $k_0^*(s,t)$ are obtained for $s>t$. Substituting these into Eqs. (25) or (26), both equations become identical, and we find that $n^{(0)}$ must satisfy the condition

$$J(t)=\hbar\,\partial_t n^{(0)}(t). \quad (31)$$

This gives the EOM for $n^{(0)}$, and we can solve this to express $n^{(0)}$ as a functional of J, with the result

$$n^{(0)}[t;J]=n^{(0)}(t_1)+\int_{t_1}^{t}ds\,\frac{J(s)}{\hbar}. \quad (32)$$

Thus Eqs. (29), (30), and (32) determine the unperturbed propagator G_0 with structure (16), where h and g are replaced by h_0 and g_0, respectively.

As already seen from Eqs. (32) or (31), we succeeded in making the expectation value of the number depend on J in the leading order, i.e., $O(\lambda^0)$. This makes the inversion formula applicable. The right-hand side of Eq. (32) corresponds to the desired lowest-order functional $f^{(0)}$ in Eq. (4), and Eq. (31) is the inverted relation, the right-hand side of which corresponds to $g^{(0)}$ of Eq. (5). So our next task is to calculate the perturbative correction to n, and then to derive the correction to the EOM [Eq. (31)] with the aid of the inversion formulas.

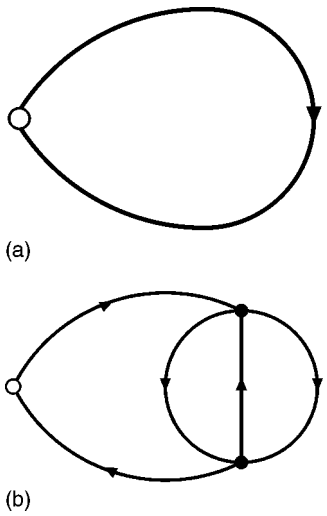

FIG. 1. Diagrams for the expectation value $n[J]$. (a) The leading order $O(\lambda^0)$. (b) The next-to-leading order $O(\lambda^2)$. The vertex of the open circle expresses $\psi_1^*\psi_1$ at t.

C. Perturbative correction and the QKE

With the propagator $-G_0$ expressed by the arrow going from ψ^* to ψ and the vertex $\pm\lambda/i\hbar$ (the signs + and −, respectively, correspond to the forward and backward time paths), the unperturbed number $n_{\mathbf{k}}^{(0)}$ is represented diagrammatically as Fig. 1(a), and the nonzero perturbative correction to $n_{\mathbf{k}}[t;J]$ first comes from a diagram shown in Fig. 1(b), which is of $O(\lambda^2)$.

The contributions of $O(\lambda)$ from a tadpole type self-energy insertion to Fig. 1(a) vanish, because those from the vertices on forward and backward time branches cancel each other. In general, the contributions from the diagrams with tadpoles do not vanish for higher orders, but can be renor-

malized into the one-particle energy ϵ by the constant shift

$$\epsilon_{\mathbf{k}}^{R}=\epsilon_{\mathbf{k}}+\lambda\sum_{\mathbf{q}} n_{\mathbf{q}}^{(0)R}, \tag{33}$$

and by preparing the corresponding counterterm. [The superscript R in the right-hand side of Eq. (33) expresses that $n^{(0)}$ is written in terms of the renormalized ϵ.] In the following, this renormalization will be understood without the superscript R, and diagrams with tadpoles will not be considered.

As the result, the expectation value of the number as a functional of J is given up to the next-to-leading order as

$$n_{\mathbf{k}}[t,J]=n_{\mathbf{k}}^{(0)}[t,J]+\left(\frac{\lambda}{\hbar}\right)^2\sum_{\mathbf{q},\mathbf{k}'}\int_{t_I}^{t}dt'\int_{t_I}^{t'}ds' \times\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t'-s')\}\tilde{N}^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s'), \tag{34}$$

where

$$\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}\equiv\frac{1}{\hbar}(\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{k}'-\mathbf{k}}-\epsilon_{\mathbf{q}}-\epsilon_{\mathbf{k}'-\mathbf{q}}), \tag{35}$$

$$\tilde{N}^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t)\equiv(n_{\mathbf{k}}^{(0)}(t)+1)(n_{\mathbf{k}'-\mathbf{k}}^{(0)}(t)+1)n_{\mathbf{q}}^{(0)}(t)n_{\mathbf{k}'-\mathbf{q}}^{(0)}(t) -n_{\mathbf{k}}^{(0)}(t)n_{\mathbf{k}'-\mathbf{k}}^{(0)}(t)(n_{\mathbf{q}}^{(0)}(t)+1)(n_{\mathbf{k}'-\mathbf{q}}^{(0)}(t)+1). \tag{36}$$

Recall that $\tilde{N}^{(2)}$ is a functional of J since, all $n^{(0)}$'s in Eq. (36) are functionals of J given in Eq. (32). Equation (34) corresponds to $f_0+\lambda f_1+\lambda^2 f_2$ of Eq. (4), where f_1 vanishes as mentioned above.

Applying the inversion formula (10), the correction to EOM (31) is obtained. In this case, on the right-hand side of Eq. (10), $f_0^{(2)}$ and $f_1^{(1)}$ vanish and $f_0^{(1)^{-1}}(t,s)$ is $\hbar\partial_t\delta(t-s)$. Thus we obtain the $O(\lambda^2)$ term by operating $-\hbar\partial_t\delta(t-s)$ to the second term of the right-hand side of (34), and by replacing $n^{(0)}[t;J]$ contained in $\tilde{N}^{(2)}$ by $n(t)$; In course of the inversion, as seen from Eq. (7), all the functionals of J are evaluated at $J_{\mathbf{k}}=\hbar\dot{n}_{\mathbf{k}}$, and since the functional expression of $n^{(0)}[J]$ is given by Eq. (32), $n^{(0)}[J]$'s in the functional are replaced by $n(t)$. Thus we obtain the inversion of Eq. (34) as

$$J_{\mathbf{k}}(t)=\hbar\partial_t n_{\mathbf{k}}(t)-\frac{\lambda^2}{\hbar}\sum_{\mathbf{q},\mathbf{k}'} \times\int_{t_1}^{t}ds\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t'-s')\}N^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s), \tag{37}$$

where $N^{(2)}$ is defined by Eq. (36), in which all the $n^{(0)}$'s are replaced by n. If we set the external source $J=0$, Eq. (37) gives the EOM for the number which is nothing but the QKE. This type of QKE is known as a non-Markovian extension of the Boltzmann equation (see, e.g., Refs. [16–18]) and, as we will see in Sec. III E, the total energy (sum of the kinetic and interaction energies) is conserved by this QKE.

It is expected that dissipative effect appears from the summation over wave numbers, and hence the collisional information at time s decays with some time scale. If this time scale is much shorter than that of the variation of the number, the QKE can be approximated by a Markovian equation, which is obtained by the adiabatic expansion. Setting the initial time $t_I=-\infty$, we expand $N^{(2)}(s)$ around the time t as $N^{(2)}(s)=N^{(2)}(t)+(s-t)\dot{N}^{(2)}(t)+\cdots$, regarding the time differentiations to be small. Then the integral becomes

$$\int_{-\infty}^{t}ds\cos\omega(t-s)N^{(2)}(s)=\pi\delta(\omega)N^{(2)}(t) +\frac{\wp}{\omega^2}\dot{N}^{(2)}(t)+\cdots, \tag{38}$$

where $\wp$ expresses the principal value. The second term of the right-hand side is proportional to $\dot{n}$, and gives a perturbative correction to the coefficient of the first term on the right-hand side of Eq. (37), which becomes $O(\lambda^4)$ and can be neglected (cf., however, Refs. [16,18] for its meaning.) Regarding all higher time derivatives to be small, we take into account up to the first term of Eq. (38), and obtain the familiar Boltzmann equation

$$\hbar\partial_t n_{\mathbf{k}}(t)=\pi\lambda^2\sum_{\mathbf{q},\mathbf{k}'}\delta(\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{k}-\mathbf{k}'}-\epsilon_{\mathbf{q}}-\epsilon_{\mathbf{q}-\mathbf{k}'})N^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}. \tag{39}$$

As is well known, with this Markovian Boltzmann equation, the conservation of the total energy is not realized, and only the kinetic energy is conserved, and in the sense that the interaction do not contribute to nondissipative characteristics of the system, the Boltzmann equation [Eq. (39)] is referred to as the kinetic equation of ideal gas [19].

D. Higher-order corrections

The higher-order correction to the QKE is quite systematically derived by the inversion method. Let us see the next-order correction to QKE (37), i.e., the $O(\lambda^3)$ term, in the framework presented in Sec. III B. (Of course the higher orders in the framework of Appendix B can be calculated in the same way.) We first calculate the next order of Eq. (34), and then the corresponding correction to the EOM is derived by inversion formulas.

The corrections of $O(\lambda^3)$ to $n[t;J]$ come from the diagrams shown in Fig. 2. They are evaluated as

$$\begin{aligned}n_{\mathbf{k}}^{(3)}[t;J]=\left(\frac{\lambda}{\hbar}\right)^3\sum_{\mathbf{k}',\mathbf{q},\mathbf{q}'}\int_{t_1}^{t}dt'\int_{t_1}^{t'}ds\int_{t_1}^{s}ds' &\times\{\tfrac{1}{2}\sin\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t'-s)+\omega_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s-s')\}\\ &\times\tilde{N}^{(1)}_{\mathbf{q},\mathbf{k}'}(s)\tilde{N}^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s')\\ &-\tfrac{1}{2}\sin\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t'-s)+\omega_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(s-s')\}\\ &\times\tilde{N}^{(1)}_{\mathbf{k},\mathbf{k}'}(s)\tilde{N}^{(2)}_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(s')\\ &+2\sin\{\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t'-s)+\omega'_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s-s')\}\\ &\times\tilde{M}^{(1)}_{\mathbf{q},\mathbf{k}'}(s)\tilde{M}^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s')\\ &-2\sin\{\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t'-s)+\omega'_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(s-s')\}\\ &\times\tilde{M}^{(1)}_{\mathbf{k},\mathbf{k}'}(s)\tilde{M}^{(2)}_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(s')\},\end{aligned} \tag{40}$$

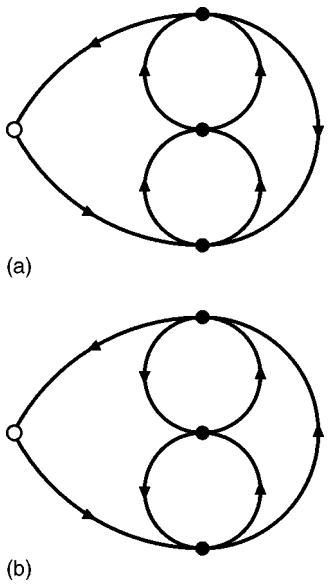

FIG. 2. Diagrams for $O(\lambda^3)$ contributions to $n[J]$.

where ω and $N^{(2)}$ are defined in Eqs. (35) and (36), respectively, and

$$\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'} \equiv \frac{1}{\hbar}(\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{q}-\mathbf{k}'}-\epsilon_{\mathbf{q}}-\epsilon_{\mathbf{k}-\mathbf{k}'}) = \omega_{\mathbf{k},\mathbf{q},\mathbf{k}+\mathbf{q}-\mathbf{k}'}, \tag{41}$$

$$\tilde{N}^{(1)}_{\mathbf{k},\mathbf{k}'} \equiv (n^{(0)}_{\mathbf{k}}+1)(n^{(0)}_{\mathbf{k}'-\mathbf{k}}+1)-n^{(0)}_{\mathbf{k}}n^{(0)}_{\mathbf{k}'-\mathbf{k}'}, \tag{42}$$

$$\tilde{M}^{(1)}_{\mathbf{k},\mathbf{k}'} \equiv (n^{(0)}_{\mathbf{k}}+1)n^{(0)}_{\mathbf{k}-\mathbf{k}'}-n^{(0)}_{\mathbf{k}}(n^{(0)}_{\mathbf{k}-\mathbf{k}'}+1), \tag{43}$$

$$\begin{aligned}\tilde{M}^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'} &\equiv (n^{(0)}_{\mathbf{k}}+1)(n^{(0)}_{\mathbf{q}'-\mathbf{k}'}+1)n^{(0)}_{\mathbf{k}-\mathbf{k}'}n^{(0)}_{\mathbf{q}'} \\ &\quad -n^{(0)}_{\mathbf{k}}n^{(0)}_{\mathbf{q}'-\mathbf{k}'}(n^{(0)}_{\mathbf{k}-\mathbf{k}'}+1)(n^{(0)}_{\mathbf{q}'}+1) \\ &= \tilde{N}^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}+\mathbf{q}'-\mathbf{k}'}.\end{aligned} \tag{44}$$

Note that $\tilde{N}$ and $\tilde{M}$ are functionals of J, since $n^{(0)}$'s in these expressions are given as Eq. (32). Equation (40) corresponds to $f_3[t;J]$ of Eq. (4).

The correction of $O(\lambda^3)$ to the EOM of n is obtained by the third order inversion formula

$$\begin{aligned}g_3[t;Q] = -\int dt' f_0^{(1)^{-1}}[t,t';Q]\Big(&\tfrac{1}{6}\int dsds'ds'' \\ &\times f_0^{(3)}[t',s,s',s'';Q]g_1[s;Q]g_1[s';Q]g_1[s'';Q] \\ &+\int dsds' f_0^{(2)}[t',s,s';Q]g_1[s;Q]g_2[s';Q] \\ &+\tfrac{1}{2}\int dsds' f_1^{(2)}[t',s,s';Q]g_1[s;Q]g_1[s';Q] \\ &+\int ds f_1^{(1)}[t',s;Q]g_2[s;Q] \\ &+\int ds f_2^{(1)}[t',s;Q]g_1[s;Q]+f_3[t';g_0[Q]]\Big),\end{aligned} \tag{45}$$

with the notation in Sec. II B. As in the case of $O(\lambda^2)$, only the last term in the parentheses on the right-hand side of Eq. (45) makes a nonzero contribution, since f_0 is linear in J and $f_1=g_1=0$. Thus the inversion of Eq. (40) is obtained as

$$\begin{aligned}J_{\mathbf{k}}(t) = \hbar\partial_t n_{\mathbf{k}}(t) &- \frac{\lambda^2}{\hbar}\sum_{\mathbf{q},\mathbf{k}'}\int_{t_I}^{t} ds\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)\}N^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s) \\ &-\frac{\lambda^3}{\hbar^2}\sum_{\mathbf{k}',\mathbf{q},\mathbf{q}'}\int_{t_I}^{t}ds\int_{t_I}^{s}ds' \\ &\times\{\tfrac{1}{2}\sin\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)+\omega_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s-s')\} \\ &\times N^{(1)}_{\mathbf{q},\mathbf{k}'}(s)N^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s') \\ &-\tfrac{1}{2}\sin\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)+\omega_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(s-s')\} \\ &\times N^{(1)}_{\mathbf{k},\mathbf{k}'}(s)N^{(2)}_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(s') \\ &+2\sin\{\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s) \\ &+\omega'_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s-s')\} \\ &\times M^{(1)}_{\mathbf{q},\mathbf{k}'}(s)M^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s') \\ &-2\sin\{\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)+\omega'_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(s-s')\} \\ &\times M^{(1)}_{\mathbf{k},\mathbf{k}'}(s)M^{(2)}_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(s')\},\end{aligned} \tag{46}$$

where N and M are now functions of n which are respectively obtained from $\tilde{N}$ and $\tilde{M}$ by replacing $n^{(0)}[J]$'s with n's as in Eq. (37). Then the QKE with $O(\lambda^3)$ corrections is obtained by setting $J=0$ in Eq. (46). We will see in Sec. III E that the total energy is conserved by QKE (46), similarly as in Eq. (37).

There appear new collision terms in Eq. (46) whose forms are quite different from the usual Boltzmann-type collision factor. The first two terms in the braces of Eq. (46) are contributions from the diagram in Fig. 2(a), and are due to the binary collision. The last two terms are from the diagram in Fig. 2(b) which expresses the three-body collision effect. These will be seen from the diagrammatic structure of Fig. 2, or from the fact that, in the dilute limit, the first two terms become $O(n^2)$, while the last two terms become $O(n^3)$. Let us see these more closely.

In the first two terms, the factor $N^{(2)}$ is the ordinary Boltzmann-type collision factor and $N^{(1)}$ has a ''gain-loss'' form found by Morozov and Röpke [4] in the study of the initial correlation effect; we will see a similar term in Sec. IV, where we discuss the initial correlation [see Eq. (89)]. In Ref. [4], a generalized binary T-matrix approximation is considered, and, after the Born approximation, the gain-loss factor appears in one of the initial correlation terms. There [and in Eq. (89)], the correlations at the initial stage is multiplied by the gain-loss factor, while in Eq. (46) the binary collision at the earlier time is multiplied by it. This suggests that the gain-loss factor appears from the collision of a correlated pair of particles. In Eq. (46) the correlation take place in the preliminary collision at time s', which provides the factor $N^{(2)}(s')$, whereas in Ref. [4] (or in Sec. IV) it comes from the initial correlation.

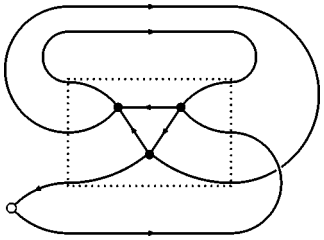

FIG. 3. Diagram of the three-body collision term. The subdiagram surrounded by the dotted line expresses the three-body collision.

The last two terms in the braces of Eq. (46) look slightly more complicated. The fact that these terms are contributions from the three-body collision can be seen by rewriting Fig. 2(b) as Fig. 3. The part of the diagram encircled by the dotted line represents the three-body scattering. The factor $M^{(2)}$ is the usual binary collision factor, but the factor $M^{(1)}$ has the form of the one-body process, and does not appear in the binary collision approximation. This will be the time to see such a contribution in the QKE.

The factor $M^{(1)}$ can be interpreted as coming from the collision between a correlated particle, which has collided with some other particle before, and another particle which has been moving freely: For example, considering the diagram of Fig. 4(a), where the vertices are arranged in temporal order from right to left, we can rewrite it as Fig. 4(b) by replacing each vertex with a local interaction via the dotted line. (The replacement is not unique.) If we focus on the arrows going from right to left in Fig. 4(b), the particle propagating with a single line is a free one in the sense that it has not collided before, and a double line represents the correlated particle. Then the one-body factor $M^{(1)}$ appears from the point denoted by a double circle, where a single line changes into a double one, and we can see that the free particle collides there with a correlated particle.

The adiabatic expansion of Eq. (46) becomes somewhat complicated, but can be carried out similarly as in Eq. (38). In the third term of Eq. (46), the first term in the braces, for example, becomes

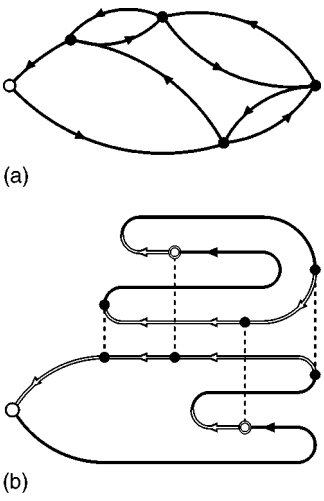

FIG. 4. The appearance of the one-body factor $M^{(1)}$.

$$\frac{\lambda^3}{2\hbar^2}\sum_{\mathbf{k}',\mathbf{q},\mathbf{q}'}\left(\left\{-\frac{\wp}{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}}+\pi\delta'(\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'})\partial_t\right\}N^{(1)}_{\mathbf{q},\mathbf{k}'}(t)\right.$$
$$\times\left\{\pi\delta(\omega_{\mathbf{k},\mathbf{q}',\mathbf{k}'})+\frac{\wp}{\omega^2_{\mathbf{k},\mathbf{q}',\mathbf{k}'}}\partial_t\right\}N^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(t)$$
$$+\left\{\pi\delta(\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'})+\frac{\wp}{\omega^2_{\mathbf{k},\mathbf{q},\mathbf{k}'}}\partial_t\right\}N^{(1)}_{\mathbf{q},\mathbf{k}'}(t)$$
$$\left.\times\left\{-\frac{\wp}{\omega_{\mathbf{k},\mathbf{q}',\mathbf{k}'}}+\pi\delta'(\omega_{\mathbf{k},\mathbf{q}',\mathbf{k}'})\partial_t\right\}N^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(t)\right). \quad (47)$$

Similar expressions are obtained for other terms. The terms with the time derivative give higher-order corrections as in the case of Eq. (38), and the terms without the time derivative represent the correction to the long time behavior of the QKE. Thus the QKE is reduced to the Markovian equation as

$$\hbar\partial_t n_{\mathbf{k}}(t)=\frac{\pi\lambda^2}{\hbar}\sum_{\mathbf{q},\mathbf{k}'}\delta(\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'})N^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t)-\frac{2\lambda^3}{\hbar^2}\sum_{\mathbf{k}',\mathbf{q},\mathbf{q}'}$$
$$\times\left\{\tfrac{1}{4}\,\mathrm{Im}\left(\frac{1}{\omega^-_{\mathbf{k},\mathbf{q},\mathbf{k}'}}\frac{1}{\omega^-_{\mathbf{k},\mathbf{q}',\mathbf{k}'}}\right)N^{(1)}_{\mathbf{q},\mathbf{k}'}N^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(t)\right.$$
$$-\tfrac{1}{4}\,\mathrm{Im}\left(\frac{1}{\omega^-_{\mathbf{k},\mathbf{q},\mathbf{k}'}}\frac{1}{\omega^-_{\mathbf{q}',\mathbf{q},\mathbf{k}'}}\right)N^{(1)}_{\mathbf{k},\mathbf{k}'}N^{(2)}_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(t)$$
$$+\mathrm{Im}\left(\frac{1}{\omega'^-_{\mathbf{k},\mathbf{q},\mathbf{k}'}}\frac{1}{\omega'^-_{\mathbf{k},\mathbf{q}',\mathbf{k}'}}\right)M^{(1)}_{\mathbf{q},\mathbf{k}'}M^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(t)$$
$$\left.-\mathrm{Im}\left(\frac{1}{\omega'^-_{\mathbf{k},\mathbf{q},\mathbf{k}'}}\frac{1}{\omega'^-_{\mathbf{q}',\mathbf{q},\mathbf{k}'}}\right)M^{(1)}_{\mathbf{k},\mathbf{k}'}(t)M^{(2)}_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(t)\right\}, \quad (48)$$

where we have used the abbreviation $1/\omega^-\equiv 1/(\omega-i0)=\wp/\omega+i\pi\delta(\omega)$. This Markovian expression coincides with the one derived in Ref. [20], where the tadpole contributions, which are renormalized as Eq. (33) in our theory, are explicitly calculated. It is shown there that, in the Markovian case, the second and fourth terms in the braces of Eq. (48) can be transformed into the first and third terms, respectively, and thus the right-hand side of Eq. (48) can be expressed as $N^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}$ multiplied by the modified amplitude. But due to this transformation, the three-body aspects of the collision term are lost in Ref. [20].

It is not difficult to see that this Markovian equation conserves only the kinetic energy, and that the Bose-Einstein distribution for an ideal gas is a stationary solution of Eq. (48). Thus, like the Markovian Boltzmann equation (39), the interaction does not contribute to the equilibrium property, although the higher-order corrections of the collision are included in Eq. (48). In this sense, Eq. (48) is a proper extension of the Boltzmann equation as the kinetic equation of ideal gas as in Sec. III C.

The three-body T-matrix approximation to the QKE was studied in several works [21], and, according to Ref. [21], a three-body collision factor like $n^3(n+1)^3-(n+1)^3n^3$ is expected to appear together with the three-particle energy conservation factor after the Markovian approximation. From

our observations, however, such a collision term will not appear in the QKE if the elementary interaction is of a two-body nature. The collision at each time produces at most a factor $N^{(2)}(M^{(2)})$, $N^{(1)}$, or $M^{(1)}$, or a factor like $(n+1)^2n-n^2(n+1)$ (which does not appear in the approximation presented here). After the adiabatic expansion, the collision term can be expressed by time-local product of these factors, but a factor like $n^3(n+1)^3-(n+1)^3n^3$ does not appear from such a product.

The appearance of such a three-body collision term in Ref. [21] is due to the formal double-time expression of the three-body T matrix; of the six time arguments in T matrix $T(t_1,t_2,t_3;t_1',t_2',t_3')$, the incoming three and outgoing three are equated and represented as $T(t;t')$, respectively. This is formally realized by writing the three-body interaction as the product of two-body interaction and unity (expressing non-interaction): $V_{123}=v_{12}I_3+v_{23}I_1+v_{31}I_2$. If the three propagators going into (out of) the T matrix really end at (start from) the same vertex and hence contain the numbers at the same instant, a three-body collision factor like $n^3(n+1)^3-(n+1)^3n^3$ can be produced. But this is not the case if the elementary interaction is binary. In each term of the perturbative expansion, only two of the three propagators going into (out of) the T matrix end at (start from) the same vertex v and carry numbers at the same instant, while the other one does so with numbers at different times. Thus the three-body collision term considered in Ref. [21] does not appear from an elementary interaction of binary nature.

E. Conservation of the total energy

When we set $J=0$ the expectation value of the total Hamiltonian is time independent. The conservation of the total energy by the QKE derived above can easily be seen from a direct calculation of the time derivatives of the kinetic energy and the interaction energy. Below, we show that these cancel each other order by order.

First in $O(\lambda^2)$, the time derivative of the kinetic energy $E_{\rm kin}\equiv\langle\hat{H}_0\rangle$ is calculated by multiplying QKE (37) (setting $J=0$) by $\epsilon_{\mathbf{k}}$, and summing over $\mathbf{k}$, with the result

$$\partial_t E_{\rm kin}(t)=\left(\frac{\lambda}{\hbar}\right)^2\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\int_{t_{\rm I}}^{t}ds\,\frac{\hbar\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}}{4}\times\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)\}N^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s). \tag{49}$$

Then, in the same order of the approximation, the interaction energy $E_{\rm int}\equiv\langle\hat{H}_{\rm int}\rangle$ is calculated from the diagrams shown in Figs. 5(a) and 5(b). At first, with the propagator appearing in Sec. III B, it is calculated as a functional of J; then, by evaluating it at $J=\hbar\dot{n}$, we obtain the interaction energy written in terms of n. As the result, the first-order contribution [Fig. 5(a)] gives a constant

$$E^{(1)}_{\rm int}=\frac{\lambda}{2}\left(\sum_{\mathbf{q}}n_{\mathbf{q}}\right)^2, \tag{50}$$

and the time-dependent contribution comes from Fig. 5(b), which is evaluated as

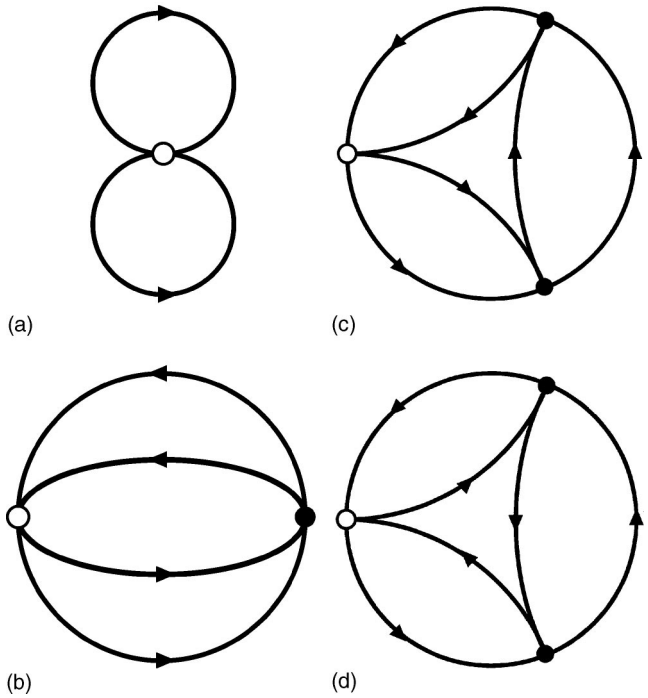

FIG. 5. Diagrams for the interaction energy. The vertex of the open circle expresses $(\lambda/4)\psi_1^*\psi_1^*\psi_1\psi_1$.

$$E^{(2)}_{\rm int}(t)=-\frac{\lambda^2}{4\hbar}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\int_{t_{\rm I}}^{t}ds\,\sin\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)\}N^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s). \tag{51}$$

Apparently, the time derivative of Eq. (51) cancels with Eq. (49), and hence the total energy is conserved in $O(\lambda^2)$.

Similarly, the correction of $O(\lambda^3)$ to the time derivative of the kinetic energy [Eq. (49)] is calculated from the third term of Eq. (46) as

$$\begin{aligned}&\left(\frac{\lambda}{\hbar}\right)^3\sum_{\mathbf{k},\mathbf{k}',\mathbf{q},\mathbf{q}'}\int_{t_{\rm I}}^{t}ds\int_{t_{\rm I}}^{s}ds'\\&\times\left\{\frac{\hbar\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}}{4}\sin\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)+\omega_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s-s')\}\right.\\&\times N^{(1)}_{\mathbf{q},\mathbf{k}'}(s)N^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s')\\&+\hbar\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'}\sin\{\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)+\omega'_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s-s')\}\\&\left.\times M^{(1)}_{\mathbf{q},\mathbf{k}'}(s)M^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s')\right\},\end{aligned} \tag{52}$$

and the correction to the interaction energy is calculated from the diagrams shown in Figs. 5(c) and 5(d), which lead to the result

$$\begin{aligned}E^{(3)}_{\rm int}(t)=&-\frac{\lambda^3}{\hbar^2}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q},\mathbf{q}'}\int_{t_{\rm I}}^{t}ds\int_{t_{\rm I}}^{s}ds'\\&\times\{\tfrac{1}{4}\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)+\omega_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s-s')\}\\&\times N^{(1)}_{\mathbf{q},\mathbf{k}'}(s)N^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s')\\&+\cos\{\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)+\omega'_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s-s')\}\\&\times M^{(1)}_{\mathbf{q},\mathbf{k}'}(s)M^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s')\}.\end{aligned} \tag{53}$$

We can easily see that the time derivative of Eq. (53) precisely cancels with Eq. (52). This is seen by separating the time derivative of Eq. (53) into two parts: a differentiation with respect to t in the integrand, and one with respect to t in the upper bound of the integral; the former cancels with Eq. (52), while the latter vanishes by the symmetry on exchanging $\mathbf{k}$ and $\mathbf{q}'$. This order-by-order cancellation of the time derivative of the total energy is an interesting and welcome feature in the case of using the source [Eq. (20)], and does not hold in the case of another source in Appendix B.

As pointed out before, the Markovian QKE's (39) or (48), approximated versions of Eqs. (37) or (46), respectively, do not conserve the total energy but conserve only the kinetic energy. This is in agreement with the former observations [16,17,19] that the total energy conservation is essentially connected with the memory effect.

IV. INITIAL CORRELATIONS

Let us consider the initial correlation effects which were neglected in Sec. III. The inclusion of initial correlations was investigated in Refs. [22,11], and formulated as a perturbation theory with a 3×3 matrix form propagator in Refs. [13,12], or as an initial condition for the Martin-Schwinger hierarchy in Ref. [23]. In this section, we show that our theory can include initial correlations following the treatment of Refs. [13,12,4]. Introducing the imaginary-time path in addition to the forward and backward time paths, we calculate the propagator on this complex time path, and perform a perturbative calculation in terms of the interaction and the initial correlation. The resultant expressions are compared with those in Ref. [4], and the stationarity in the case of initial equilibrium is confirmed.

A. Description of initial state and mixed propagator

The initial state is assumed to be homogeneous, and is described by the density matrix

$$\hat{\rho}(t_{\mathrm{I}})=(\mathrm{Tr}\, e^{-\beta\hat{S}})^{-1}e^{-\beta\hat{S}}, \tag{54}$$

where $\hat{S}$ is a function of $\hat{\psi}$ and $\hat{\psi}^{\dagger}$ (see Ref. [24] for the details of constructing $\hat{\rho}$). The factor β is just for notational convention and need not be related to equilibrium temperature. (In the case of initial equilibrium, $\hat{S}$ is replaced by $\hat{H}$, and β expresses the inverse temperature.) Decomposing $\hat{S}$ into quadratic term $\hat{S}_0$ and higher-order terms $\hat{S}_{\mathrm{corr}}$, we assume the form

$$\hat{S}=\hat{S}_0+\hat{S}_{\mathrm{corr}}=\sum_{\mathbf{k}}\varsigma_{\mathbf{k}}\hat{\psi}^{\dagger}_{\mathbf{k}}\hat{\psi}_{\mathbf{k}}+\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\tilde{\lambda}\,\hat{\psi}^{\dagger}_{\mathbf{k}-\mathbf{q}}\hat{\psi}^{\dagger}_{\mathbf{k}'+\mathbf{q}}\hat{\psi}_{\mathbf{k}}\hat{\psi}_{\mathbf{k}'}, \tag{55}$$

where $\tilde{\lambda}$ expresses the strength of the initial correlation. In the following we assume $\tilde{\lambda}$ to be small, and consider perturbative corrections in $\tilde{\lambda}$. Of course more general types of correlations can be treated in a similar manner as far as $\hat{S}_{\mathrm{corr}}$ is allowed to be regarded as a perturbation.

With the initial density [Eq. (54)], the matrix element $\langle\psi_{1\mathrm{I}}|\hat{\rho}_{\mathrm{I}}|\psi_{2\mathrm{I}}\rangle$ of (2) can be represented by the usual coherent-state path integral as

$$\langle\psi_{1\mathrm{I}}|\hat{\rho}_{\mathrm{I}}|\psi_{2\mathrm{I}}\rangle\propto\int[d\psi_3]e^{(i/\hbar)\int_0^{\beta\hbar}d\tau(i\hbar\Sigma_{\mathbf{k}}\psi^{*}_{3,\mathbf{k}}\psi_{3,\mathbf{k}}+iS(\psi_3))}, \tag{56}$$

where ψ_3 is introduced as an integration variable which satisfies the boundary conditions $\psi_3(\beta\hbar)=\psi_{1\mathrm{I}}$ and $\psi_3(0)=\psi_{2\mathrm{I}}$, and $S(\psi_3)$ is obtained from $\hat{S}$ by replacing the operator $\hat{\psi}$ and $\hat{\psi}^{\dagger}$ in Eq. (55) by ψ and ψ^{*}, respectively. As usual, ψ_3 evolves in imaginary time from $\tau=0$ to $\tau=\beta\hbar$ with the evolution operator $\hat{S}$, and hence the time path is extended to include this imaginary time. Hereafter, we use Greek letters to express the imaginary time.

Then, as in Refs. [12], [4], the Green function is extended to 3×3 matrix form,

$$\tilde{G}\equiv-\mathrm{Tr}\,\hat{\rho}\begin{pmatrix}T\hat{\psi}(t)\hat{\psi}^{\dagger}(s) & \hat{\psi}^{\dagger}(s)\hat{\psi}(t) & \hat{\psi}^{\dagger}(\sigma)\hat{\psi}(t)\\ \hat{\psi}(t)\hat{\psi}^{\dagger}(s) & \tilde{T}\hat{\psi}(t)\hat{\psi}^{\dagger}(s) & \hat{\psi}^{\dagger}(\sigma)\hat{\psi}(t)\\ \hat{\psi}(\tau)\hat{\psi}^{\dagger}(s) & \hat{\psi}(\tau)\hat{\psi}^{\dagger}(s) & T_{\tau}\hat{\psi}(\tau)\hat{\psi}^{\dagger}(\sigma)\end{pmatrix}_c$$
$$=\begin{pmatrix}G(t,s) & \tilde{k}(t,\sigma)\\ & \tilde{k}(t,\sigma)\\ \tilde{h}(\tau,s)\tilde{h}(\tau,s) & \tilde{g}(\tau,\sigma)\end{pmatrix}, \tag{57}$$

where $\hat{\psi}(\tau)\equiv e^{(\tau/\hbar)\hat{S}}\hat{\psi}e^{(-\tau/\hbar)\hat{S}}$ and T_{τ} is the imaginary-time ordering. The real-time component $G(t,s)$ is given in Eq. (16), and the imaginary-time one is

$$\tilde{g}(\tau,\sigma)=\theta(\tau-\sigma)\,\eta(\tau,\sigma)+\theta(\sigma-\tau)\,\kappa(\tau,\sigma), \tag{58}$$

with

$$\eta(\tau,\sigma)\equiv-\langle\hat{\psi}(\tau)\hat{\psi}^{\dagger}(\sigma)\rangle_c, \tag{59}$$

$$\kappa(\tau,\sigma)\equiv-\langle\hat{\psi}^{\dagger}(\sigma)\hat{\psi}(\tau)\rangle_c, \tag{60}$$

and the mixed-time parts are defined by

$$\tilde{h}(\tau,s)\equiv-\langle\hat{\psi}(\tau)\hat{\psi}^{\dagger}(s)\rangle_c, \tag{61}$$

$$\tilde{k}(t,\sigma)\equiv-\langle\hat{\psi}^{\dagger}(\sigma)\hat{\psi}(t)\rangle_c. \tag{62}$$

Then the unperturbed propagator $\tilde{G}_0$ under the source J can be obtained as the inverse of

$$\tilde{\mathcal{D}}=\begin{pmatrix}i\hbar\partial_t-\epsilon+iJ(t) & -iJ(t) & 0\\ -iJ(t) & -i\hbar\partial_t+\epsilon+iJ(t) & 0\\ 0 & 0 & i\hbar\partial_{\tau}+i\varsigma\end{pmatrix}. \tag{63}$$

As in Sec. III B, assuming that $\tilde{G}_0$ has the similar structure as Eq. (57), in which h, g, etc. are replaced by h_0, g_0, etc., respectively, we solve the inverse relation

$$\tilde{\mathcal{D}}\tilde{G}_0=\tilde{G}_0\tilde{\mathcal{D}} \tag{64}$$

$$=-i\hbar\begin{pmatrix}\delta(t-s) & 0\\ 0\ \ 0 & \delta(\tau-\sigma)\end{pmatrix}. \tag{65}$$

Apparently for the real-time component $G(t,s)$, the equations for h_0 and k_0 are the same as Eqs. (23)–(28), and hence the solution is the same.

For the imaginary time element $\tilde{g}_0$, Eq. (65) leads to the equations

$$(\hbar\partial_\tau+\varsigma)\eta_0(\tau,\sigma)=0 \quad (\text{for } \tau>\sigma), \tag{66}$$

$$(\hbar\partial_\tau+\varsigma)\kappa_0(\tau,\sigma)=0, \quad (\text{for } \sigma>\tau), \tag{67}$$

with the boundary condition at $\tau=\sigma$.

$$\eta_0(\sigma,\sigma)-\kappa_0(\sigma,\sigma)=-1. \tag{68}$$

This can be easily solved, with the results

$$\eta_0(\tau,\sigma)=-(n^{(0)}(t_1)+1)e^{-(\varsigma/\hbar)(\tau-\sigma)}, \tag{69}$$

$$\kappa_0(\tau,\sigma)=-n^{(0)}(t_1)e^{(\varsigma/\hbar)(\sigma-\tau)}. \tag{70}$$

Finally for the mixed-time sector, Eq. (65) gives

$$(\hbar\partial_\tau+\varsigma)\tilde{h}_0(\tau,s)=0, \tag{71}$$

$$(i\hbar\partial_t-\epsilon)\tilde{k}_0(t,\sigma)=0, \tag{72}$$

and the boundary conditions are given from the conditions $\psi_1(t_1)=\psi_3(\beta\hbar)$ and $\psi_2(t_1)=\psi_3(0)$, which yield

$$\tilde{h}_0(\tau,t_1)=\kappa_0(\tau,\beta\hbar)=\eta_0(\tau,0), \tag{73}$$

$$\tilde{k}_0(t_1,\sigma)=\eta_0(\beta\hbar,\sigma)=\kappa_0(0,\sigma), \tag{74}$$

$$\tilde{h}_0(0,s)=h_0^*(s,t_1), \quad \tilde{h}_0(\beta\hbar,s)=k_0^*(s,t_1), \tag{75}$$

$$\tilde{k}_0(t,0)=k_0(t,t_1), \quad \tilde{k}_0(t,\beta\hbar)=h_0(t,t_1). \tag{76}$$

Then, solving Eqs. (71) and (72) as

$$\tilde{h}_0(\tau,s)=-e^{-(\varsigma/\hbar)\tau}\tilde{h}_0(0,s), \tag{77}$$

$$\tilde{k}_0(t,\sigma)=-e^{-(i/\hbar)\epsilon(t-t_1)}\tilde{k}_0(t_1,\sigma), \tag{78}$$

the boundary conditions (73)–(76) are satisfied by

$$\tilde{h}_0(0,s)=-e^{(i/\hbar)\epsilon(s-t_1)}(n^{(0)}(t_1)+1), \tag{79}$$

$$\tilde{k}_0(t_1,\sigma)=-n^{(0)}(t_1)e^{(\varsigma/\hbar)\sigma}, \tag{80}$$

with the initial unperturbed number

$$n^{(0)}(t_1)=\frac{1}{e^{\beta\varsigma}-1}. \tag{81}$$

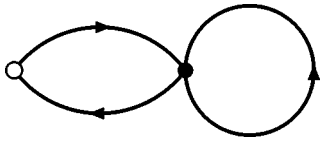

FIG. 6. The tadpole self-energy inserted diagram of $O(\tilde{\lambda})$.

Summarizing, the real-time part is given by Eqs. (30), (29), and (32) with the initial value of Eq. (81), and that of the imaginary-time component by Eqs. (69) and (70), and the mixed parts

$$\tilde{h}_0(\tau,s)=-e^{-(\varsigma/\hbar)\tau}(n^{(0)}(t_1)+1)e^{(i/\hbar)\epsilon(s-t_1)}, \tag{82}$$

$$\tilde{k}_0(t,\sigma)=-e^{-(i/\hbar)\epsilon(t-t_1)}n^{(0)}(t_1)e^{(\varsigma/\hbar)\sigma}. \tag{83}$$

As discussed in Ref. [6], with the appropriate counter-term, it is possible to make $n^{(0)}$ in η_0 or κ_0 depend on imaginary time, but this is not necessary in our problem.

B. Initial correlations in the QKE

With the above derived unperturbed propagator, let us calculate the expectation value of the number operator up to first order in the initial correlation and second order in the interaction. This time, the diagrams are constructed with the propagator $\tilde{G}_0$ calculated above and with vertices which come from $\hat{H}_{\rm int}$, if it is on the real-time axis, and from $\hat{S}_{\rm corr}$, if it is on the imaginary time axis.

In contrast to the case in Sec. III, the tadpole self-energy inserted diagram does not vanish if the vertex is in imaginary time. Thus, in $O(\tilde{\lambda})$, the initial correlation effect appears from Fig. 6, which simply gives a constant

$$\Delta n_{\mathbf{k}}=-\beta\tilde{\lambda}n_{\mathbf{k}}^{(0)}(t_1)(n_{\mathbf{k}}^{(0)}(t_1)+1)\sum_{\mathbf{q}} n_{\mathbf{q}}^{(0)}(t_1). \tag{84}$$

This expresses a shift of the initial value due to the initial correlation, but such a constant term is not convenient for the inversion method since $n^{(0)}[J=\hbar\dot{n}]$ does not simply gives n. To avoid this inconvenience, similarly as in Eq. (33), we renormalize the quadratic part of $\hat{S}$ by

$$\varsigma_{\mathbf{k}}^R=\varsigma_{\mathbf{k}}+\tilde{\lambda}\sum_{\mathbf{q}} n_{\mathbf{q}}^{(0)R}(t_1), \tag{85}$$

and introduce a corresponding counterterm to cancel the tadpole. As in Sec. III, the superscript R will be suppressed in the following.

The correction of $O(\lambda\tilde{\lambda})$ to $n_{\mathbf{k}}(t)$ comes from the diagram of the same form as Fig. 1(b), but this time one vertex is on the real-time axis, which expresses $\hat{H}_{\rm int}$, and the other one is on the imaginary-time axis, representing $\hat{S}_{\rm corr}$. As a result, after integration over imaginary time,

$$\frac{\lambda\tilde{\lambda}}{\hbar^2}\sum_{\mathbf{q},\mathbf{k}'}\frac{\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-t_1)\}-1}{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}}\frac{\tilde{N}^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t_1)}{\Omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}} \tag{86}$$

is obtained, where $\Omega_{\mathbf{k},\mathbf{q},\mathbf{k}'} \equiv 1(\varsigma_{\mathbf{k}}+\varsigma_{\mathbf{k}'-\mathbf{k}}-\varsigma_{\mathbf{q}}-\varsigma_{\mathbf{k}'-\mathbf{q}})/\hbar$. The contributions of $O(\lambda^2\tilde{\lambda})$ come from the diagrams in Fig. 2, with one of the vertices on the imaginary-time axis. Then they are calculated as

$$\frac{\lambda^2\tilde{\lambda}}{\hbar^3}\sum_{\mathbf{k}',\mathbf{q},\mathbf{q}'}\int_{t_I}^{t}ds\int_{t_I}^{s}ds' \times\Bigg\{ \tfrac{1}{2}\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s-s')+\omega_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s'-t_I)\} \times\tilde{N}^{(1)}_{\mathbf{q},\mathbf{k}'}(s')\frac{\tilde{N}^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'},(t_I)}{\Omega_{\mathbf{k},\mathbf{q}',\mathbf{k}'}} - \tfrac{1}{2}\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s-s')+\omega_{\mathbf{q}',\mathbf{q},\mathbf{k}}(s'-t_I)\} \times\tilde{N}^{(1)}_{\mathbf{k},\mathbf{k}'}(s')\frac{\tilde{N}^{(2)}_{\mathbf{q},\mathbf{q},\mathbf{k}'}(t_I)}{\Omega_{\mathbf{q}',\mathbf{q},\mathbf{k}'}} + 2\cos\{\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s-s')+\omega'_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s'-t_I)\} \times\tilde{M}^{(1)}_{\mathbf{q},\mathbf{k}'}(s')\frac{\tilde{M}^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(t_I)}{\Omega'_{\mathbf{k},\mathbf{q}',\mathbf{k}'}} - 2\cos\{\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s-s')+\omega'_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(s'-t_I)\} \times\tilde{M}^{(1)}_{\mathbf{k},\mathbf{k}'}(s')\frac{\tilde{M}^{(2)}_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(t_I)}{\Omega'_{\mathbf{q}',\mathbf{q},\mathbf{k}'}}\Bigg\}, \tag{87}$$

where $\Omega'_{\mathbf{k},\mathbf{q}',\mathbf{k}'} \equiv (1)(\varsigma_{\mathbf{k}}+\varsigma_{\mathbf{q}'-\mathbf{k}'}-\varsigma_{\mathbf{q}'}-\varsigma_{\mathbf{k}-\mathbf{k}'})/\hbar$.

Since the number as a functional of source J is expressed by a double series expansion as $f=\Sigma_{n,n'}\lambda^n\tilde{\lambda}^{n'}f_{nn'}$, the inversion formulas should also be extended to double expansion: $g=\Sigma_{m,m'}\lambda^m\tilde{\lambda}^{m'}g_{mm'}$. Although we do not set down their somewhat lengthy expressions here, the derivation of the inversion formulas can be carried out in the same way as in Sec. II B. In our problem, since f_{10} vanishes and f_{00} is linear in J [f_{10} and f_{00} are f_1 and f_0 in Eq. (4), respectively], the formulas for $(m,m')\neq(0,0)$ up to $(m,m')=(2,1)$, are simply reduced to the form

$$g_{mm'}[t;Q]=-\int dt' f^{(1)^{-1}}_{00}[t,t';Q]f_{mm'}[t';g_{00}]. \tag{88}$$

Thus as the result of the inversion, the QKE up to $O(\lambda^2\tilde{\lambda})$ is obtained as

$$\begin{aligned}\hbar\partial_t n_{\mathbf{k}}(t)=&\frac{\lambda^2}{\hbar}\sum_{\mathbf{q},\mathbf{k}'}\int_{t_I}^{t}ds\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)\}N^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s)-\frac{\lambda\tilde{\lambda}}{\hbar}\sum_{\mathbf{q},\mathbf{k}'}\sin\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-t_I)\}\frac{N^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t_I)}{\Omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}}\\ &+\frac{\lambda^2\tilde{\lambda}}{\hbar^2}\sum_{\mathbf{k}',\mathbf{q},\mathbf{q}'}\int_{t_I}^{t}ds\Bigg\{\tfrac{1}{2}\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)+\omega_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s-t_I)\}N^{(1)}_{\mathbf{q},\mathbf{k}'}(s)\frac{N^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(t_I)}{\Omega_{\mathbf{k},\mathbf{q}',\mathbf{k}'}}\\ &-\tfrac{1}{2}\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)+\omega_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(s-t_I)\}N^{(1)}_{\mathbf{k},\mathbf{k}'}(s)\frac{N^{(2)}_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(t_I)}{\Omega_{\mathbf{q}',\mathbf{q},\mathbf{k}'}}\\ &+2\cos\{\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)+\omega'_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(s-t_I)\}M^{(1)}_{\mathbf{q},\mathbf{k}'}(s)\frac{M^{(2)}_{\mathbf{k},\mathbf{q}',\mathbf{k}'}(t_I)}{\Omega'_{\mathbf{k},\mathbf{q}',\mathbf{k}'}}\\ &-2\cos\{\omega'_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)+\omega'_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(s-t_I)\}M^{(1)}_{\mathbf{k},\mathbf{k}'}(s)\frac{M^{(2)}_{\mathbf{q}',\mathbf{q},\mathbf{k}'}(t_I)}{\Omega'_{\mathbf{q}',\mathbf{q},\mathbf{k}'}}\Bigg\}.\end{aligned} \tag{89}$$

The first term on the right-hand side is the usual binary collision term and the last two terms are a manifestation of initial correlation effects. Comparing these initial correlation terms with the collision terms in Eq. (46), we can see that the collision factors at s' in Eq. (46) are replaced by initial correlation factors which have the forms of binary collisions at the initial time t_I. The second term, which is the term linear in the interaction, commonly appears in a calculation including an initial correlation [25,4]. In the third term represented by braces, the first two terms are ''gain-loss'' terms, which were first found by Morozov and Röpke [4]. As mentioned in Sec. III D, the gain-loss factor $N^{(1)}$ appears from the collision of two particles which are initially correlated. The last two terms has the same ''one-body'' factor as Eq. (46). Although these are not genuine three-body collision term, they can be interpreted as a collision among the initially correlated two particles and another particle. To the author's knowledge, such terms have not been found before, and appear for the first time in this paper. These terms are of $O(n^3)$, and in the nondilute case, we must include these corrections in the second order of the initial correlation.

In the case of initial equilibrium, we set $\tilde{\lambda}=\lambda$ and $\varsigma=\epsilon$, and then, for $O(\lambda^2)$ in Eq. (89), the first two terms cancel each other if $N^{(2)}(s)$ in the first term is replaced by the initial value $N^{(2)}(t_I)$ which is consistent with the vanishing of the time derivative. Hence the stationarity of the equilibrium state is confirmed up to $O(\lambda^2)$. For $O(\lambda^3)$ in the case of the initial equilibrium, we need an extra correction of $O(\lambda\tilde{\lambda}^2)$ to

QKE (89), and, after somewhat tedious calculations, we can show that this initial correlation term, together with the $O(\lambda^2\tilde{\lambda})$ term in Eq. (89) cancels the $O(\lambda^3)$ collision term in Eq. (46) in the same way as $O(\lambda^2)$. Note that in these calculations, the corrections of $O(\tilde{\lambda}^2)$ and $O(\tilde{\lambda}^3)$ which contribute to constant shifts Δn, are renormalized to ς by Eq. (85) similarly to the tadpole contribution, and this renormalized ς is equated with ϵ in the initial equilibrium case.

V. SUMMARY AND DISCUSSION

We have investigated the QKE in the framework of CTP formalism. By the inversion method, with a type of probe introduced in Eq. (20), the QKE was simply derived as the EOM of the expectation value of the number operator. It should be emphasized that what we have calculated, e.g., in Eqs. (34) and (40), is just the expectation value of the number. We have made no ansatz concerning the form of the propagator, and hence there is no uncertainty in the definition of the number appearing in the QKE.

The QKE obtained by our method coincides with that derived by the GKB formalism [3], at least up to the approximation considered here. In the GKB formalism, the QKE is derived from the Dyson equation for the two-point Green function, and a closed equation for the occupation number is obtained with the use of the GKB ansatz in the form of the propagator. Roughly, the collision integral appears from the self-energy, with both ends contracted by a propagator in this formalism. The fact that our formalism gives the same result as the GKB formalization can be seen from the form of propagator derived in Sec. III B. The GKB ansatz with the free-particle approximation for retarded and advanced Green functions is recovered from the propagator in Sec. III B if we evaluate it at $J=\hbar\dot{n}$ to replace $n^{(0)}[J]$ by n, as is indeed done in the course of inversion. Up to the approximation in this paper, collision terms are calculated from diagrams in which both ends of the self-energy inserted propagator are connected to $\psi^*\psi$; after the inversion, this provides the same collision term as the GKB formalism.

Note, however, that the GKB ansatz is just an ansatz on the form of propagator, and strictly speaking, the number that appears therein is not a well defined object. On the other hand, in our method, the number appearing in the QKE is really the expectation value of the bare number operator.

For the sake of definite expressions, we have restricted ourselves to simple perturbation up to third order, and hence the obtained QKE's are special cases of those, e.g., in Refs. [16–18]. For more realistic situations where effects like dynamical screening or strong collision should be considered, a partial summation of the polarization [26,27] or ladder diagrams [11,16,18,27] will be required. These are beyond our scope, but will be treated similarly as in the GKB formalism. Inclusion of the self-energy effects will be realized by the counterterm method [5,7,8], where a part of the self-energy is renormalized into the free part of the Lagrangian, and enables a description of the quasiparticle number. It is not difficult to combine the inversion method with the counterterm method, though the approximations made there become somewhat ambiguous.

As we have seen in Sec. III A, there are some choices in the introduction of the probing source. As a general prescription, we should first introduce the source such that the expectation value $Q[J]$ is guaranteed to be real. This will restrict the form of the source to a few candidates. [In our case, ϵ, α, and γ in Eq. (A5), if the counterterm is assumed not to contain time derivatives.] Then, hopefully, the source dependence in the lowest order of the perturbation is required to be able to use the inversion formulas. (α and γ are retained in our case.) If arbitrariness is still there, a convenient form will be chosen which makes the calculation simple, guarantees the conservation law, etc.

The contents of Eq. (20) [or Eq. (B1)] becomes somewhat clear from the ''physical'' representation [15] of the CTP effective action for ψ, which is a Legendre transformation of the generating functional W, with ψ itself as the order parameter. The physical representation is introduced through the transformation of the variable $J_C \equiv \frac{1}{2}(J_1+J_2)$, and $J_\Delta \equiv J_1 - J_2$. Then physically sensible situation $J_1=J_2$ is realized by $J_\Delta=0$, and thereby J_C plays the role of a physical external source J. From the CTP generating functional W, in which the source J couples to ψ, the effective action $\Gamma[\psi_\Delta,\psi_C]$ is calculated through the Legendre transformation of $W[J_C,J_\Delta]$ [15,10], where $\psi_\Delta \equiv \delta W/\delta J_C$ and $\psi_C \equiv \delta W/\delta J_\Delta$. Roughly speaking, $\psi_\Delta=\psi_1-\psi_2$, $\psi_C=\frac{1}{2}(\psi_1+\psi_2)$ and the inverse propagator $\mathcal{D}$ is the tree part of the second derivative of Γ.

With these prescriptions, a source of the form of Eq. (20) couples to $\psi_\Delta^*\psi_\Delta$, and this means that the system is disturbed from the external by shifting $\delta^2\Gamma/\delta\psi_\Delta^*\delta\psi_\Delta$ which is the one-particle-irreducible amputated part of the correlation function $\langle\{\hat{\psi}^\dagger,\hat{\psi}\}\rangle$. This may be the reason why we can handle a number with this source. Another source discussed in Appendix B couples to $\psi_\Delta^*\psi_C-\psi_C^*\psi_\Delta$, and corresponds to the shift of the imaginary part of the retarded self-energy $\delta^2\Gamma/\delta\psi_\Delta^*\delta\psi_C$. As is discussed in Appendix B, though this source also handles the number, it gives a less useful expression for the QKE in the sense that the order-by-order conservation of the total energy shown in Sec. III E is not realized in this case.

The way to introduce the source presented here will be extended to cases other than that of the number operator, i.e., generic composite operators $Q(\hat{\varphi})$ of any dynamical variable $\hat{\varphi}$. The above discussion suggests that we can couple the source to $Q(\varphi_\Delta)$, which means a probe through $Q(\delta/\delta\varphi_\Delta)\Gamma[\varphi_\Delta,\varphi_C]$. Since the derivative of Γ with respect to φ_Δ's expresses the one-particle-irreducible amputated part of symmetrized correlation function, the source coupled to $Q(\varphi_\Delta)$ is always physically sensible, and the expectation value of $Q(\hat{\varphi})$ depends on this source nontrivially in the absence of interaction. Thus the first and second of the above mentioned criteria for the source are inherent in this choice of the source. In some cases, calculation will be simplified compared with the usual procedure with the source coupled to $Q(\varphi_1)-Q(\varphi_2)$.

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APPENDIX A: STRUCTURE OF THE DISSIPATIVE COUNTERTERM

In the original paper of Lawrie [5], the system discussed is a relativistic scalar boson field. Although the essential part of the treatment is the same for a nonrelativistic boson field, in order to make this paper self-contained here we present a derivation of the dissipative counterterm [Eq. (19)]. The starting point is a Green function with the structure of Eq. (16); keeping this structure, we construct an operator $\mathcal{D}$ which satisfies relation (22).

First we consider the case where the system is temporary homogeneous. Then the propagator become a function of the time difference; with the four real functions u, v, w, and z, we write it as

$$h(t)=u(t)+iv(t), \quad k(t)=w(t)+iz(t), \tag{A1}$$

which leads to a Fourier transformation with the form

$$\begin{aligned}\check{G}_{ij}(\omega)&\equiv\int_{-\infty}^{\infty}dt\,e^{-i\omega t}G_{ij}(t)\\&=\begin{pmatrix}A+C+i(B+D) & 2C\\ 2A & A+C-i(B+D)\end{pmatrix}\\&\quad+\begin{pmatrix}(b-d)+i(a-c) & -2d\\ -2b & (b-d)-i(a-c)\end{pmatrix}.\end{aligned} \tag{A2}$$

Here the real functions A, B, C, and D and a, b, c, and d are defined as the real and imaginary parts of the Fourier-Laplace transformations of u, v, w, and z, respectively,

$$A(\omega)+ia(\omega)\equiv\int_0^{\infty}dt\,e^{-i\omega t}u(t), \tag{A3}$$

etc., A, B, C, and D are even functions of ω, and a, b, c, and d are odd in ω.

Then the Fourier transform of $\mathcal{D}$ is obtained from the inverse relation (22), and we expand it in terms of ω. Taking up to the linear terms of ω, we require that the linear part coincides with the bare $\mathcal{D}_0$. Thus the structure of $\mathcal{D}$ is determined as

$$\check{\mathcal{D}}=\begin{pmatrix}-\hbar(\omega+\Omega)+i\alpha & i(\hbar\gamma-\alpha)\\ -i(\hbar\gamma+\alpha) & \hbar(\omega+\Omega)+i\alpha\end{pmatrix}, \tag{A4}$$

where Ω, γ, and α are real constants.

In a temporary inhomogeneous case, like an ansatz of the inverse propagator, we allow the coefficients in Eq. (A4) to be time dependent, and write $\mathcal{D}$ as

$$\mathcal{D}(t,\partial_t)=\begin{pmatrix}i\hbar\partial_t-(\hbar\Omega(t)-i\alpha(t)) & i(\hbar\gamma(t)-\alpha(t))\\ -i(\hbar\gamma(t)+\alpha(t)) & -i\hbar\partial_t+(\hbar\Omega(t)+i\alpha(t))\end{pmatrix}, \tag{A5}$$

where Ω, γ, and α are now real functions of t. Thus the dissipative counterterm matrix is defined as Eq. (19), with $\Delta\omega=\Omega-\epsilon/\hbar$. This ansatz is confirmed to be consistent with Eq. (22), and a dissipative propagator can be derived with a time-dependent number. Note that from this dissipative counterterm, unlike in the case of relativistic boson field in Ref. [5], Eq. (22) can be solved rigorously, and we do not need approximations for the unperturbed dissipative propagator.

APPENDIX B: ANOTHER TYPE OF PROBE

As mentioned in Sec. III A, there is another choice of probe which couples to $\psi_2^{\dagger}\psi_1-\psi_1^{\dagger}\psi_2$. This is the source utilizing γ in Eq. (19), and is introduced using the following inverse propagator:

$$\mathcal{D}(t,\partial_t)=\begin{pmatrix}i\hbar\partial_t-\epsilon & iJ(t)\\ -iJ(t) & -i\hbar\partial_t+\epsilon\end{pmatrix}. \tag{B1}$$

We demonstrate that the unperturbed number also shows a nontrivial dependence on this source, and that the inversion method leads to another kinetic equation, which reduces to the ordinary Boltzmann equation in the adiabatic limit.

The bare propagator in this scheme is calculated by the inverse of Eq. (B1) as in Sec. III B. The solution of Eq. (22), in which $\mathcal{D}$ is replaced by Eq. (B1) is obtained similarly as in Sec. III B. With $n^{(0)}(t)=k_0(t,t)$ satisfying the EOM

$$J(t)=-\frac{\hbar}{2}\partial_t\ln(1+2n^{(0)}(t)), \tag{B2}$$

the unperturbed propagator is given by

$$\begin{aligned}k_0(t,s)&=-n^{(0)}(s)e^{-(1/\hbar)\int_s^t dt'(\epsilon-iJ)}\\&=-n^{(0)}(s)\left(\frac{1+2n^{(0)}(t)}{1+2n^{(0)}(s)}\right)^{1/2}\\&\quad\times e^{-(i/\hbar)\epsilon(t-s)},\end{aligned} \tag{B3}$$

$$\begin{aligned}h_0(t,s)&=-(n^{(0)}(s)+1)e^{-(i/\hbar)\int_s^t dt'(\epsilon-iJ)}\\&=-(n^{(0)}(s)+1)\left(\frac{1+2n^{(0)}(t)}{1+2n^{(0)}(s)}\right)^{1/2}\\&\quad\times e^{-(i/\hbar)\epsilon(t-s)},\end{aligned} \tag{B4}$$

with the matrix structure of Eq. (16). Equation (B2) can be solved for $n^{(0)}$ as

$$n^{(0)}[t;J]=\{(\tfrac{1}{2}+n^{(0)}(t_1))e^{-(2/\hbar)\int_{t_1}^t dsJ(s)-\frac{1}{2}}\}. \tag{B5}$$

The correction to Eq. (B5) appears from the same diagram as Fig. 1(b), but with the above derived propagator. Then the result is

$$\begin{aligned}n_{\mathbf{k}}[t,J]&=n_{\mathbf{k}}^{(0)}(t)+\left(\frac{\lambda}{\hbar}\right)^2\sum_{\mathbf{q},\mathbf{k}'}\\&\quad\times\int_{t_I}^t dt'\frac{1+2n_{\mathbf{k}}^{(0)}(t)}{1+2n_{\mathbf{k}}^{(0)}(t')}\int_{t_I}^{t'}ds'\left\{\frac{\tilde{P}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t')}{\tilde{P}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s')}\right\}^{1/2}\\&\quad\times\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t'-s')\}\tilde{N}^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s'),\end{aligned} \tag{B6}$$

where

$$\tilde{P}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t)=(1+2n^{(0)}_{\mathbf{k}})(1+2n^{(0)}_{\mathbf{k}'-\mathbf{k}}) \times(1+2n^{(0)}_{\mathbf{q}'})(1+2n^{(0)}_{\mathbf{k}'-\mathbf{q}'})(t). \quad \text{(B7)}$$

Applying the inversion formulas, the correction to Eq. (B2) is obtained, and the source J is expressed by n as

$$J_{\mathbf{k}}(t)=-\frac{\hbar}{2}\partial_t \ln(1+2n_{\mathbf{k}}(t)) + \frac{\lambda^2}{\hbar(1+2n_{\mathbf{k}}(t))}\sum_{\mathbf{q},\mathbf{k}'}\int_{t_I}^{t} ds\left\{\frac{P_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t)}{P_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s)}\right\}^{1/2} \times\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)\}N^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s), \quad \text{(B8)}$$

where P is defined from Eq. (B7) by replacing $n^{(0)}$'s with n's. Then removing the source in Eq. (B8), and rearranging it for $\partial_t n_{\mathbf{k}}$, the EOM is obtained as

$$\hbar\partial_t n_{\mathbf{k}}=\frac{\lambda^2}{\hbar}\sum_{\mathbf{q},\mathbf{k}'}\int_{t_I}^{t} ds\left\{\frac{P_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t)}{P_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s)}\right\}^{1/2} \times\cos\{\omega_{\mathbf{k},\mathbf{q},\mathbf{k}'}(t-s)\}N^{(2)}_{\mathbf{k},\mathbf{q},\mathbf{k}'}(s). \quad \text{(B9)}$$

Comparing Eq. (B9) with the QKE obtained in Sec. III C, we see that they differ by the factor $\{P(t)/P(s)\}^{1/2}$. The adiabatic expansion of Eq. (B9), however, leads to the ordinary Boltzmann equation. As in Eq. (38), the integration over s in Eq. (B9) is approximated by

$$\int_{-\infty}^{t} ds\cos\{\omega(t-s)\}\frac{N^{(2)}(s)}{P^{1/2}(s)} = \pi\delta(\omega)\frac{N^{(2)}(t)}{P^{1/2}(t)}+\frac{\wp}{\omega^2}\partial_t\frac{N^{(2)}(t)}{P^{1/2}(t)}+\cdots. \quad \text{(B10)}$$

The factor $1/P^{1/2}(t)$ in the first term on the right-hand side cancels with $P^{1/2}(t)$ in Eq. (B9), and the terms including the time derivative can be neglected similarly as in Sec. III C. Thus a Boltzmann equation identical to Eq. (39) is obtained on a long-time scale.

As pointed out in Sec. II B, it is not surprising that the resultant expressions of the QKE presented here and the one in Sec. III C are different. It seems that the probe of the form of Eq. (20) is more suited for deriving a kinetic equation, since, in addition to the calculational simplicity, it guarantees the conservation of total energy at least up to the third order of the interaction, as shown in Sec. III E. Unlike in Sec. III E, the time derivatives of the kinetic and interaction energies do not completely cancel each other if we use QKE (B9). Of course the nonzero contribution to the time derivative of the total energy is of higher order in λ; this is not a serious shortcoming, but it will be useful if the time derivative of the total energy precisely cancels order by order.

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